MATH 8100: Linear Optimization

Extreme Points and BFS

polyhedron: the intersection of a finite collection of half-spaces and hyperplanes

polytope: a bounded polyhedron

Note: The feasible set of any LP is a polyhedron

$$\{x \in \mathbb{R}^n | Ax \geq b\}$$

Note: Any LP can be described as

$$\begin{aligned} \min\ & c^T x \\ & Ax \geq b \end{aligned}$$

extreme point: Given a convex set S, a point $x \in S$ is an extreme point if there does NOT exist two DISTINCT points $y, z \in S$ and $\lambda \in (0,1)$ s.t. $x = \lambda y + (1-\lambda)z$

BFS: $\bar{x}$ is BFS if $\bar{x} \in P$ and is a BS

BS: $\bar{x}$ is a BS if

- satisfies all equality constraints
- at least n of the active constraints of P at $\bar{x}$ are linearly independent (i.e. coefficients of variables are LI)

Note: $\bar{x}$ is a BS iff $\text{rank}(A_I) = n$ where I are the indices of active constraints

Theorem: $P \in \mathbb{R}^n$ a polyhedron. x is an extreme point of $P \iff x$ is a BFS.

Corollary: Given a finite number of inequality constraints, there can only be a finite number of BFS.

Existance and Optimality of Extreme Points

Note: $P \in \mathbb{R}^n$ contains a line if

$$\exists x \in P, d \in \mathbb{R}^n \setminus \{0\} \text{ s.t. } x + \lambda d \in P, \forall \lambda \in \mathbb{R}$$

Theorem: If a nonempty polyhedron does NOT contain a lines $\iff$ it has at least one extreme point.

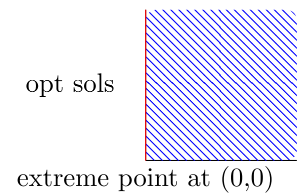
Corollary: Any polytope and any polyhedron in the form of either

$$P = \{x \in \mathbb{R}^n | Ax \geq b, x \geq 0\} \quad \text{or} \quad Q = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$$

have at least one extreme point

Theorem: Consider the LP $\min c^T x$ s.t. $x \in P$ where $P \subseteq \mathbb{R}^n$ is a polyhedron. If the LP is solvable and P has at least one extreme point, then there exists an optimal solution which is an extreme point.

Note: An extreme point is optimal, but not all optimal solutions are extreme points



Theorem: Consider the LP $\min c^T x$ s.t. $x \in P$ where $P \in \mathbb{R}^n$ is a polyhedron that has at least one extreme point. Then, either the LP is unbounded below or there exists an extremem point that is optimal.

Fundamental Theorem of LP: Suppose that P is a nonempty polyhedron s.t. $P \in \{x \in \mathbb{R}^n | x \geq 0\}$. Consider the LP $\min c^T x$ s.t. $x \in P$. If the LP is solvable, then there exists an extreme point in P that is an optimal solution to the LP.

Note: The Fundamental Theorem of LP require that $P \subseteq \mathbb{R}^n_+$. So, $P = \{x | Ax \geq b\}$ may not be suitable. That's why we do standard form.

Polyhedra in Standard Form

Standard Form of a Polyhedra: $S = \{x | Ax = b, x \geq 0\}$
Standard Form LP:

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

- **Case 1: Max Problem**
  1. change max to $-\min$
  2. negative $c^T x$ to become $-c^T x$
- **Case 2: Inequality Constraints**
  1. For $\leq$, add a slack variable.
  $$\begin{gathered} x_1 + x_2 \leq 10 \\ x_1 + x_2 + x_3 = 10; x_3 \geq 0 \end{gathered}$$
  2. For $\geq$, first negate entire inequality, then add slack variable
  $$\begin{gathered} x_1 + x_2 \geq 8 \\ -x_1 - x_2 \leq 8 \\ -x_1 - x_2 + x_3 = 8; x_3 \geq 0 \end{gathered}$$
- **Case 3: Free variable** (not listed as $x_i \geq 0$)
  1. For free variable, x_i, let $x_i = x_i^+ - x_i^-$
  2. Replace x_i by $x_i^+ - x_i^-$ where $x_i^+, x_i^- \geq 0$

Basic Solutions in Standard Form: Suppose $A \in \mathbb{R}^{m \times n}$ has full row rank, and $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ is a nonempty polyhedron. x is a BS $\iff$ x solves $Ax = b$ and there exists column indicies $B(1), \cdots, B(m)$ s.t.

- The columns $A_{B(1)}, \cdots, A_{B(m)}$ of A are LI
- If $j \notin \{B(1), \cdots, B(m)\}$, then $x_j = 0$

basic/nonbasic variables: For a basic solution, variables $x_{B(1)}, \cdots, x_{B(m)}$ are basic variables, the remainin are nonbasic variables.
basic columns: The columns $A_{B(1)}, \cdots, A_{B(m)}$ are the basic columns and form a basis in $\mathbb{R}^m$.
set of basic indices: $\{B(1), \cdots, B(m)\}$

Polyhedra in Standard Form Cont.

Note:

- nonbasic variables must be 0. basic variables can be 0.
- $Ax = b$ can be written as $\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$;
  where $B = \begin{bmatrix} A_{B(1)} & \cdots & A_{B(m)} \end{bmatrix}$
  $x_M = \begin{bmatrix} x_{B(1)} & \cdots & x_{B(m)} \end{bmatrix}^T$; and $x_N = 0$
- $Bx_B = b \implies x_B = B^{-1} b$
- If x is a BFA, then $x \geq 0$ and $x_B = B^{-1} b \geq 0$

Optimality Conditions of Extreme Points

A standard form LP can have the KKT conditions applied to it. KKT is necessary (LCQ) and sufficient (convex) for optimality.

$$\begin{aligned} (PF)\ & x_B = B^{-1} b \geq 0 \\ & x_N = 0 \\ (DF)\ & \lambda_B, \lambda_N \geq 0 \\ & c_b - \lambda_B + B^T \mu = 0 \\ & c_N - \lambda_N + N^T \mu = 0 \\ (CS)\ & \lambda_i x_i = 0; i = 1, \cdots, n \end{aligned}$$

Theorem: Suppose that x is a BFS with basis matrix B and define $\bar{c}$ by

$$\bar{c}^T = c^T - c_B^T B^{-1} A$$

- if $\bar{c} \geq 0$, then x is optimal
- if x is optimal with positive basic variables ($x_B > 0$), then $\bar{c} \geq 0$

reduced cost: $\bar{c}$ is the vector of reduced cost. for each j, $\bar{c}_j$ is the reduced cost of x_j.
Note: $\bar{c} = \begin{bmatrix} c_B \\ C_N \end{bmatrix} - \begin{bmatrix} B^T \\ N^T \end{bmatrix} (B^T)^{-1} c_B$
Note: If x_j is a basic variable, $\bar{c}_j = 0$
optimal: A basic matrix B is said to be optimal if $B^{-1} b \geq 0$ and $\bar{c} \geq 0$ where $\bar{c}$ is the reduced cost.
basic direction:

$$\begin{aligned} & d_B = -B^{-1} A_j \\ & d_j = 1 \\ & d_i = 0 \ \ \text{(for all nonbasic indices } i \neq j) \end{aligned}$$

The Simplex Method

-d	c
b	A

canonical form:

- $b \geq 0$
- A contains an identity submatrix
- the coefficients corresponding to I are all 0.

Imporant Notes:

- The variables corresponding to the identity columns with 0's as coefficients are your basic variables
- If any $b < 0$, then your solution is NOT feasible
- If any $c < 0$, then your solution is NOT optimal
- You objective value is your negative of your left corner value

Simplex Rule: How to select a pivot to decrease the objective value

1. Pick column with $c_k < 0$
2. Pick row by the minimum ratio test: the smallest ratio of b value over a value s.t. the a value is postive

$$\frac{b_n}{a_{nk}} = \min\{\frac{b_i}{a_{ik}} | a_{ik} > 0\}$$

Development of the Simplex Method

If any j s.t. $\bar{c}_j = c_j - c_B^T B^{-1} A_j < 0$ at a BFS x, then we have direction d:

$$\begin{aligned} d_B &= -B^{-1}A_j \\ d_j &= 1 \\ d_i &= 0 \text{ (for all nonbasic indices } i \neq j) \end{aligned}$$

For this d, we have that $Ad = 0$ and $c^T d < 0$ and $x + \theta d$ is feasible when θ is small and $c^T(x + \theta d) < C^T x$. So, the simplex tableau is

$-c_B^T B^{-1} b$	$\bar{c}^T = c^T - c_B^T B^{-1} A$
$B^{-1} b$	$B^{-1} A$

Theorem: Suppose that x is a BFS of a standard form LP with reduced cost $\bar{c}_j = c_j - C_B^T B^{-1} A < 0$. Consider $y = x + \theta^* d$ where

$$\begin{aligned} d_B &= -B^{-1}A_j \\ d_j &= 1 \\ d_i &= 0 \text{ (for all nonbasic indices } i \neq j) \end{aligned}$$

and $\theta^* = -\frac{x_{B(\ell)}}{d_{B_\ell}} = \min_{i=1,\cdots,m \text{ s.t. } d_{B(i)}<0} \left(-\frac{x_{B(i)}}{d_{B(i)}}\right)$ then y is a BFS associated with the basic matrix $\bar{B} = \begin{bmatrix} A_{B(1)} & \cdots & A_{B(\ell-1)} & A_{B(j)} & A_{B(\ell+1)} & \cdots & A_{B(m)} \end{bmatrix}$ and the basic indices $\{\bar{B}(1), \cdots, \bar{B}(m)\}$ where

$$\bar{B}(i) = \begin{cases} B(i) & i \neq \ell \\ j & i = j \end{cases}$$

and variable $x_{B(\ell)}$ leaves the basis and x_j enters the basis.
Note: θ^* is basically the minimum ratio test.
degenerate LP: A BFS has a value of 0.

Finding an Initial BFS

Artificial Variables:

1. First, add a as many columns to the end of your tableau as you have elements in your b vector. They will be identity in the meat of the tableau with 1's in the coefficient row.
2. do row reductions to get the coefficients for artificial variables to 0
3. do simplex method.

Note: If the original problem has a feasible solution $\bar{x}$, the artificial problem has an optimal solution $(x^*, y^*) = (\bar{x}, 0)$.
Note: If (x^*, y^*) is an optimal solution to the artificial problem with optimal value 0, then $y^* = 0$ and hence x^* is feasible for the original problem.
Theorem: That original problem is feasible iff its artificial problem has optimal value 0.
Note:

- After solving artificial problem, it optimal value is NOT 0 $\implies$ infeasible
- after getting into canonical form: if column of all negatives $\implies$ unbounded.

Big M Method:

1. Add artificial variables but make each coefficient M instead of 1.
2. do row reductions to get the coefficients for artificial variables to 0
3. do simplex method

Note: The artificial problem will never be unbounded. It could be infeasible, but never unbounded.

Duality Theory in LP

General Primal:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t}\ & a_i^T x \geq b_i,\ mi \in M_1 \\ & a_i^T x \leq b_i, \quad i \in M_2 \\ & a_i^T x = b_i, \quad i \in M_3 \\ & x_j \geq 0, \quad j \in N_1 \\ & x_j \leq 0, \quad j \in N_2 \\ & x_j \text{ free} \quad j \in N_3 \end{aligned}$$

General Dual:

$$\begin{aligned} \max\ & b^T p \\ \text{s.t.}\ & p_i \geq 0, \quad i \in M_1 \\ & \_i \leq 0, \quad i \in M_2 \\ & p_i = 0, \quad i \in M_3 \\ & A_j^T p \leq c_j, \quad j \in N_1 \\ & A_j^T p \geq c_j, \quad j \in N_2 \\ & A_j^T p = c_j, \quad j \in N_3 \end{aligned}$$

Note:

	primal	min $c^T x$	max $b^T y$	dual
A	constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
A^T	variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Note: The dual of the dual is the primal.

Theorem: If we transform the dual into an equivalent minimization problem, and then form its dual, we obtain a problem equivalent to the original primal problem.

Weak Duality Theorem: If x and p are feasible solutions to primal and dual LPs, respectively, then $c^T p \leq c^T x$

Corollary: If the primal LP is unbounded, then the dual LP is infeasible.

Corollary: If x and p are feasible solutions to the primal and dual LPs respectively, and $b^T p = c^T x$, then x and p are optimal for primal and dual respectively.

Strong Duality in LP Continued

Strong Duality Theorem: If a LP is solvable and has an optimal solution, then so is the dual, and optimal values of both problems are equal.

		primal		
		finite optimum	unbounded	infeasible
dual	finite optimum	✓	×	×
	unbounded	×	×	✓
	infeasible	×	✓	✓

Complementary Slackness Theorem: If x and p are feasible for the primal and dual problems respectively, then they are optimal for the respective problems iff

$$p_i(a_i^T x - b)i) = 0, \quad A_i$$
$$(c_i - A_j^T p)x_j = 0, \quad A_j$$

Recall the primal and dual LPs for this theorem are:

$$\begin{aligned} &\min \quad c^T x \\ &\text{s.t.} \quad Ax \geq b \\ &x \geq 0 \end{aligned}$$

$$\begin{aligned} &\max \quad b^T p \\ &\text{s.t.} \quad A^T p \leq c \\ &p \geq 0 \end{aligned}$$

Dual Simplex Method

Dual Simplex Method: With the primal problem's tableau (P) with $c \geq 0$:

1. Select h s.t. $b_h < 0$.
   - If $b \geq 0$ and (P) is in canonical form, then we have an optimal solution.
2. If $-a_{hj} \leq 0$ for all j, (D) is unbounded and hence (P) is infeasible.
3. Select k s.t $\frac{c_k}{a_{hk}} = \max\{\frac{c_j}{a_{hj}} | a_{hj} < 0\}$
4. Pivot at a_{hk} to 1.

Local Sensitivity Analysis

Goal: Analyze how an optimal solution changes with certain changes in the LP. Here, our starting tableau is M and our original optimal tableau is M^*.

Changes in available resources (b)

1. Find the Q (pivot matrix) such that $M^* = QM$
   - Q's first column is the vector e_1
   - Q remaining columns are the columns of the slack variables in M^*.
2. Change the optimval value and b column in the optimal tableau to Q times the first column in M with the change in b.
3. If not optimal, use dual simplex method.

Changes is selling price (c)

1. If the change in price is q, we simply replace the corresponding column in the M^* tableau to Q times the corresponding column from M with the change in price subtracted
2. Perform pivots to get in canonical form.
3. Determine optimality conditions

Adding new products or constraints

1. Adding a new column, n, to M means your add Qn to M^*.
2. Apply dual-simplex method.